11/1/24 Lecture Notes (4.2)

Definition: let A be an $n \times m$ matrix, we call the set of all solutions to $A\vec{x} = \vec{0}$ the nullspace of A, or null(A)

Definition: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation then the set of vectors $\vec{x}$ in $\mathbb{R}^n$ such that $T(\vec{x}) = \vec{0}$ is called the kernel of T, denoted ker(T)

- $T(\vec{x}) = A\vec{x}$, then ker(T) = null(A)

Theorem: Let $T: \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then ker(T) is a subspace of $\mathbb{R}^n$ and range(T) is a subspace of $\mathbb{R}^m$

↳ Range(T) = span of columns of A

Recall from last class: $S = \text{span}(\vec{u}_1, \dots, \vec{u}_m)$ is a subspace

S = solutions to $A\vec{x} = 0$ is a subspace

Ex: $T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} x_1 - x_2 + x_3 \\ x_1 + 2x_2 - x_3 \end{bmatrix}$ what is ker(T) and range(T)?

range$(T) \Longrightarrow T(\vec{x}) = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 2 & -1 \end{bmatrix}\vec{x} \Rightarrow \text{range}(T) = \text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$

$= \mathbb{R}^2$

ker$(T) \Rightarrow \left[\begin{array}{ccc|c} 1 & -1 & 1 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1/3 & 0 \\ 0 & 1 & -2/3 & 0 \end{array}\right] \Big\} x_1 = -\frac{1}{3}s_1,\ x_2 = \frac{2}{3}s_1,\ x_3 = s_1$

$\vec{x} = s_1 \begin{bmatrix} -1/3 \\ 2/3 \\ 1 \end{bmatrix} \Rightarrow \text{ker}(T) = \text{span}\left(\begin{bmatrix} -1/3 \\ 2/3 \\ 1 \end{bmatrix}\right)$

Theorem: T is 1-1 if and only if ker$(T) = \{0\}$

↳ Big Theorem v.4 = Big Theorem v.3 + g)

Basis and Dimension

- If S is a subspace, we can find $\vec{u}_1, \dots, \vec{u}_m$ such that S = span$(\vec{u}_1, \dots, \vec{u}_m)$

case 1: $\{\vec{0}\} = \text{span}(\vec{0})$

case 2: we have L.I. vectors whose span is S. (If $S \neq \{0\}$ and is a subspace). Let $\vec{u}$ be in S, $\vec{u}_1 \neq \vec{0}$, then the span$(\vec{u}_1)$ is a subset of S. If span$(\vec{u}_1) = S$, then you're done. If not, pick a vector in S, not in span$(\vec{u}_1)$, call it $\vec{u}_2$. Repeat until span$(\vec{u}_1, \dots, \vec{u}_m) = S$

Definition: A set $\mathcal{B} = \{u_1, \dots, u_m\}$ is a basis for a subspace S provided that:

(a) $\mathcal{B}$ spans S

(b) $\mathcal{B}$ is linearly independent

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Ex) $\left\{\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}\right\}$ is a basis in $\mathbb{R}^2$

HW: Ex) $S = \{$set of all $\begin{bmatrix}x\\y\\z\end{bmatrix} \mid x+y+z=0\}$ L.I.

if = 0, $z = s_1,\ y = s_2,\ x = -s_1 - s_2$

if 2 eq's $S = \left\{s_1\begin{bmatrix}-1\\0\\1\end{bmatrix} + s_2\begin{bmatrix}-1\\1\\0\end{bmatrix}\right\} \Rightarrow S = \text{span}\left(\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right)$

same thing, so $\left\{\begin{bmatrix}-1\\0\\1\end{bmatrix}, \begin{bmatrix}-1\\1\\0\end{bmatrix}\right\}$ is a basis for S.

can get • **Basis** = coordinate system for a subspace

eq that **Theorem**: Let $\mathcal{B} = \{u_1, \ldots, u_r\}$ be a basis for a subspace S. Then every

= 0 vector $\vec{s}$ in S can be written uniquely as a linear combo

of $s = s_1u_1 + \ldots + s_2u_r$

• Suppose S is S since $s = \text{span}(u_1, \ldots, u_r) \Rightarrow s = s_1u_1 + \ldots s_ru_r$

Conversely, if: $s = t_1u_1 + \ldots + t_ru_r$. Then: $\vec{0} = (s_1 - t_1)u_1 + \ldots + (s_r - t_r)u_r$

Since $u_1, \ldots, u_r$ are L.I. $s_1 - t_1 = 0, \ldots, s_r - t_r = 0$ so $s_1 = t_1, \ldots, s_r = t_r$

Theorem: If S is a subspace of $\mathbb{R}^n$ then every basis has the same number of vectors

• The number of vectors in a basis is called the **dimension of S**, denoted **$\dim(S)$**

• **Finding a basis**:

Ex) $S = \text{span}\left(\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\\4\end{bmatrix}, \begin{bmatrix}-2\\2\\0\\4\end{bmatrix}\right)$ } what is a basis for S? what is $\dim(S)$?

L.I. check → row reduce (= 0) and see if there's any free variables

$$\begin{bmatrix}1 & 0 & 1 & -2\\0 & 1 & 2 & 2\\1 & 0 & 3 & 0\\0 & 1 & 4 & 4\end{bmatrix} \sim \begin{bmatrix}1 & 0 & 0 & -3\\0 & 1 & 0 & 0\\0 & 0 & 1 & 1\\0 & 0 & 0 & 0\end{bmatrix}$$

Algorithm: Pivot columns give the location of basis vectors